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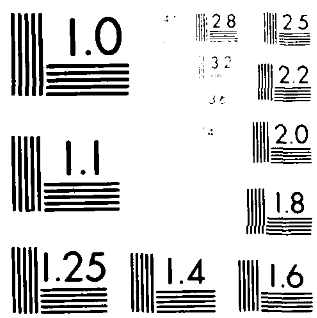
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TRANSONIC FLOW COMPUTATIONS WITH SEPARATE TREATMENT OF THE SUBSONIC AND SUPERSONIC REGIONS

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This technical report has been reviewed and is approved for publication.

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The report outlines a method for the computation of transonic flow fields in which one applies a finite element approach to the subsonic region and the method of characteristic for the supersonic region, and matches the two results at the sonic line and at the shock. The equations for the subsonic region can be obtained from Bateman's extremum principle. In each iteration step a preliminary location of the sonic line and of the shock is

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20. ABSTRACT (continued)

obtained. A new iteration step computes for this approximation of the sonic line and the given profile contour an exact supersonic flow field, and in addition, still from the supersonic region, the changes which occur in this field at the current sonic line and at the current shock location, if one changes the shape of the sonic line in a systematic manner. (In essence, one evaluates numerically the kernel of an integral which expresses the changes of the flow field in terms of the changes of the shape of the sonic line.) This information is used to formulate a boundary condition at the sonic line and at the shock in terms of the unknown changes of the shape of these two lines. With these boundary conditions a correction to the flow in the subsonic region and to the shape of the sonic line and of the shock can then be obtained by direct elimination. This procedure may be visualized as a Newton-Raphson iteration. The report discusses this procedure in general terms and derives specific formulae for the conditions at the sonic line and at the shock. The kernel mentioned above will have singularities. The nature of these singularities is explored.

FOREWORD

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The author expresses his appreciation for the efficient typing work of Miss Norma Harting and for the art work of the staff of the University of Dayton.

A

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I
INTRODUCTION

In the Murman Cole procedure the supersonic and the subsonic regions are described in the same manner. However, the difference stars for the subsonic and the supersonic regions are not the same and one needs special measures for the treatment of the sonic points and of the shock points. The strength of the Murman-Cole approach lies in the fact that these different requirements are combined into a relatively simple procedure with a minimum of logical decisions. Some of the simplicity is lost if one formulates the problems in terms of finite elements. It is still possible to retain the same elements in the subsonic and in the supersonic regions. The differences in the numerical approach for the subsonic and for the supersonic regions are brought about by the choice of weight functions. This is probably somewhat more complicated than the difference star modifications of the Murman-Cole approach. Further complications arise when one expresses the conditions at the sonic line and at the shock. The finite element procedure, in order to be economical, must use elements which are larger than one mesh of the finite difference grid system. In the Murman-Cole procedure the location of the sonic line and of the shock is only roughly defined. To leave these curves uncertain within the finite elements would impair the accuracy. A procedure of this kind has been studied elsewhere (Ref. 1).

One may ask whether it is really advantageous to use the same description for the subsonic and in the supersonic regions. The present report gives the basis for a procedure in which the supersonic and the subsonic regions are treated separately with the provision that they match at the sonic line and at the shock. At the sonic line one requires that the potential and its conormal derivative agree. At the shock one imposes the shock conditions, namely continuity of

potential and mass flow through the shock. In the present study we shall show how this can be accomplished by a Newton-Raphson procedure, in which one computes corrections to an available approximation (presumably the result of a previous iteration step). Convergence can be expected if the starting approximation is sufficiently close. In the supersonic region the flow can be computed rather rapidly by a marching procedure. It is then possible to evaluate not only the basic supersonic flow field but also variations to it, and to derive in this manner boundary conditions for the subsonic region along the common boundary of the subsonic and the supersonic regions, namely the sonic line and the shock of the (current) approximation. The resulting equations for the subsonic region (which are linear because one carries out a Newton Raphson procedure) are then solved by direct elimination. Iterations are, of course, necessary because of the nonlinearity of the problem. The iterations of the Murman-Cole procedure serve a dual purpose: they determine corrections to the flow field, and at the same time, because of immediate updating, they also take into account the nonlinearities of the problem.

The procedure just described ought to converge if one has an approximation which is sufficiently close, but it requires in each iteration step that one compute the supersonic region a number of times for different assumed shapes of the sonic line. One might think of a different approach in which the supersonic region is computed only once in each iteration step. One starts with an approximation to the subsonic field and determines the location of the sonic line. With this result one would compute the supersonic field; in particular, the values of the conormal derivative of the perturbation potential at the sonic line. For these values and with the shock conditions, corrections to the subsonic region could be computed. This would complete one iteration step.

Here one does not deal with a Newton procedure. The convergence of such iterations is uncertain even if one has a close approximation, but if the method converges, then the matching conditions at the sonic line will be satisfied. Whether one has convergence and whether the procedure is economical must be determined by numerical experiment.

SECTION II
THE SUBSONIC REGION

It is assumed that an approximation for the subsonic part of the flow field, including an approximation for the location of the sonic line and of the shock, is available. The sonic line and that part of the shock which leads to subsonic velocities constitute the common boundary between the subsonic region and the supersonic region. We refer to these data as current approximations. It is assumed that corrections to the subsonic part of the flow field are obtained by a finite element procedure. To fix the ideas we consider rectangular elements. For the present discussions which are concerned with the basic questions, the character of the shape functions to be chosen (bilinear, biquadratic, or bicubic) is unimportant. The current sonic line and the current shock cut through these elements. This must, of course, be taken into account in the computations. The subsonic problem can be formulated by means of Bateman's extremum principle; the functional whose variation must vanish is the integral of the pressure over the region under consideration. A linearization is necessary because the relation between the pressure and the velocity gradient is of a higher degree than the second. The extremum formulation is used only to derive the governing equations. The conditions at the shock and at the sonic line which occur here are not of a nature which allows one to show that the functional described above has an extremum. Convergence can be expected because one carries out a Newton-Raphson procedure if one has a sufficiently close starting approximation.

If one assumes, temporarily, that along the current sonic line and the current shock the conormal derivative to the perturbation potential is prescribed, then the potential

along this line appears as unknown; the conormal derivative appears in the equations obtained from the application of the extremum principle (which would then hold) in the inhomogeneous part. (The other contributions to the inhomogeneous part arise because in general the current approximation does not satisfy the original nonlinear equations exactly.) Actually, the conormal derivative is unknown. Computations to be carried out in the supersonic region express it and also the change of the potential along the sonic line and along the shock in terms of the change of the shape of the sonic line and of the shock. With these additional relations, one computes simultaneously the change of the potential in the subsonic part of the flow field, the change of the shape of the sonic line, and the change of the shape of the shock. The character of the conditions at the sonic line and at the shock and the computations necessary in the supersonic region will be discussed in the next sections.

SECTION III
CONDITIONS AT THE SONIC LINE

It is our goal to express the change of the potential and of its gradient in terms of some characterization of the deformation of the sonic line. We confine our attention to the supersonic region. Consider, simultaneously, a basic flow field and this basic flow field with a superimposed perturbation. Points for which the velocity vector is the same in the two fields are called corresponding points. The vector leading from a certain point of the basic flow field to its corresponding point in the perturbed field is called the displacement vector of this point. It is practical to express perturbations at the current sonic line in terms of the component of the displacement vector normal to the streamlines. (For the simplified transonic equation this is the component in the y direction.) Let s be a parameter which changes monotonically along the sonic line (in most cases the x coordinate can serve as such a parameter) and let, for some neighboring field, $d_n(s)$ and $d_t(s)$ be the components of the displacement vector for points of the sonic line, respectively, in the direction normal and tangential to the streamlines. The computations in the supersonic region serve to express $d_t(s)$ in terms of $d_n(s)$. This will be discussed later. We express the change of the gradient of the potential in terms of $d_n(s)$ and $d_t(s)$. First we compute some derivatives pertaining to the basic flow field in terms of geometric quantities. This can be done without difficulty even for the full potential equation in a local coordinate system where the temporary x axis coincides with the flow direction and the y axis is normal to it. Then one has $\phi_y = 0$ and $\phi_x = a^*$ (because we consider a point of the sonic line). The potential equation yields

$$\phi_{yy} = 0 \tag{1}$$

Let R be the radius of curvature of the streamline at the point in question. Then one has

$$\phi_{xy} = a^* \cdot \frac{1}{R} \qquad (2)$$

a* is the sonic velocity, usually one chooses a* = 1. Carrying a* within the equations facilitates a check of the dimensions. Let β be the angle of the sonic line with the local x axis. At the sonic line one has

$$\phi_x{}^2 + \phi_y{}^2 = a^{*2}$$

and since in the chosen system of coordinates $\phi_y = 0$ one obtains by differentiation with respect to x (along the sonic line)

$$\phi_x(\phi_{xx} + \phi_{xy} \text{ tg } \beta) = 0$$

Hence

$$\phi_{xx} = -\phi_{xy} \text{ tg } \beta = -a^*R^{-1} \text{ tg } \beta \qquad (3)$$

In the local system of coordinates $d_t(s)$ and $d_n(s)$ are the displacement components, respectively, in the x and y directions. The components of the gradient of the potential in the directions tangential and normal to the stream lines are given by

$$\phi_t = \phi_x \quad , \quad \phi_n = \phi_y$$

Let the perturbation potential be denoted by $\Delta\phi$. Expressing the fact that at corresponding points of the sonic line the velocity vectors in the basic and in the changed flow field are the same, and introducing a linearization one obtains

$$\Delta\phi_t + \phi_{xx}d_t + \phi_{xy}d_n = 0$$

$$\Delta\phi_n + \phi_{xy}d_t + \phi_{yy}d_n = 0$$

Hence

$$\Delta\phi_t(s) = a^*R^{-1}(\text{tg } \beta \; d_t(s) - d_n(s)) \tag{4}$$

$$\Delta\phi_n(s) = -a^*R^{-1}d_t(s) \tag{5}$$

These equations constitute boundary conditions for the perturbations in the subsonic region at the current sonic line provided that $d_t(s)$ is known as a function of $d_n(s)$. This relation will be written in the form

$$d_t(s) = \int_0^s K(s,t)d_n'(t)dt \tag{6}$$

The derivative $d_n' = d(d_n(t))/dt$ is used in order to obtain a kernel $K(s,t)$ which is less singular (and therefore better suited for a numerical evaluation). The determination of $K(s,t)$ will be shown in Section V.

SECTION IV
CONDITIONS AT A SHOCK

It will be assumed in the coming dicussions that the shock is weak. Then the change of the entropy through the shock is small, and the flow downstream of the shock is again a potential flow. At the shock the potential and the mass flow through the shock are continuous. (Continuity of the potential assures that the velocity component tangential to the shock is continuous.) Quantities referring to the flow field upstream or downstream of the shock will have, respectively, superscript 1 and 2. It is assumed that at the current shock position the basic flow field is known and that $\Delta\phi^{(1)}$, $\Delta\phi_x^{(1)}$ and $\Delta\phi_y^{(1)}$ are expressed as functions of $d_n(s)$; so far $d_n(s)$ is unknown. The basic flow field in the supersonic region will be determined for the current shape of the sonic line by means of the method of characteristics. The potential at the current shock position upstream and downstream of the shock are usually not the same because the potential $\phi^{(2)}$ is given by the current approximation to the subsonic part of the flow field, while the potential $\phi^{(1)}$ is taken from the newly computed supersonic region. The shock position in the original and in the perturbed flow fields are characterized by

$$x = x_{shock}(y) \qquad \text{and} \qquad x = x_{shock}(y) + \Delta x_{shock}(y)$$

Expressing the requirement that at the changed shock position the potential is continuous, one obtains

$$\phi^{(1)}(x_{shock},y) + \Delta\phi^{(1)}(x_{shock},y) + \phi_x^{(1)}(x_{shock},y)\Delta x_{shock}(y)$$

$$= \quad \phi^{(2)}(x_{shock},y) + \Delta\phi^{(2)}(x_{shock},y) + \phi_x^{(2)}(x_{shock},y)\Delta x_{shock}(y)$$

Hence

$$\Delta\phi^{(2)}_{shock} = \phi^{(1)}(x_{shock},y) - \phi^{(2)}(x_{shock},y) + \Delta\phi^{(1)}(x_{shock},y) +$$

$$(\phi_x^{(1)}(x_{shock},y) - \phi_x^{(2)}(x_{shock},y))\Delta x(y)_{shock} \qquad (7)$$

The only unknown quantities in the right hand side are $\Delta\phi^{(1)}$ which depends upon $d_n(s)$ and $\Delta x_{shock}(y)$. For simplicity the equations of conservation of mass for the flow through the shock are derived only for the simplified transonic equation; it expresses conservation of mass in terms of a perturbation potential which characterizes the deviation from a parallel flow with the sonic velocity. This differential equation is given by

$$-(\gamma + 1)a^{*-1}\phi_x\phi_{xx} + \phi_{yy} = 0 \qquad (8)$$

Hence

$$\iint_{\Omega} [-(\gamma + 1)a^{*-1}\phi_x\phi_{xx} + \phi_{yy}]dxdy = 0$$

and by partially carrying out the integrations

$$\int_{\partial\Omega} - [(\gamma + 1)/2\ a^{*-1}\phi_x^2\ dy + \phi_y dx] = 0$$

Hence for the deviations of the mass flow from a parallel sonic flow for a line element in the flow field with slope dx/dy

$$- [(\gamma + 1)/2\ a^{*-1}\phi_x^2 + \phi_y\ dx/dy]\ dy \qquad (9)$$

The requirement that this deviation be the same upstream and downstream of the shock gives

$$(\gamma + 1)a^{*-1}\phi_x^{(2)}\,\Delta\phi_x^{(2)} + \Delta\phi_y^{(2)}(dx_{shock}/dy)$$

$$= ((\gamma + 1)/2)a^{*-1}(\phi_x^{(1)2} - \phi_x^{(2)2}) + (\phi_y^{(1)} - \phi_y^{(2)})(dx_{shock}/dy)$$

$$+ (\gamma + 1)a^{*-1}\phi_x^{(1)}\Delta\phi_x^{(1)} + \Delta\phi_y^{(1)}(dx_{shock}/dy)$$

$$+ (\gamma + 1)a^{*-1}(\phi_x^{(1)}\phi_{xx}^{(1)} - \phi_x^{(2)}\phi_{xx}^{(2)})\Delta x_{shock}$$

$$+ (\phi_{xy}^{(1)} - \phi^{(2)})\Delta x_{shock}(dx_{shock}/dy)$$

$$+ (\phi_y^{(1)} - \phi_y^{(2)})(d\Delta x_{shock}/dy) \qquad (10)$$

The unknown quantities on the left hand side are $\Delta\phi_x(2)$ and $\Delta\phi_y^{(2)}$. (Actually the expression on the left is the conormal derivative pertaining to the differential equation for $\Delta\phi^{(2)}$). The unknown quantities on the right hand side are $\Delta x_s(y)$ and $d_n(s)$. The latter function occurs in $\Delta\phi^{(1)}$, $\Delta\phi_x^{(1)}$ and $\Delta\phi_y^{(1)}$.

SECTION V
TREATMENT OF THE SUPERSONIC REGION

For the computations of the supersonic flow field we consider the shape of the sonic line as given. In addition one knows the profile contour. In this section we shall discuss the application of the method of characteristics, although there exist finite difference methods which could also be used. In multidimensional problems finite difference methods would probably be preferable.

The method of characteristics for the full potential equation and also for the simplified transonic equation is well known. To express the ideas which are important in the present context, we shall repeat the main equations for the simplified transonic equation but without reference to the underlying mathematical theory. One starts from Eq. (8), namely

$$-(\gamma + 1)a^{*-1}\phi_x\phi_{xx} + \phi_{yy} = 0$$

This equation is rewritten in the form

$$[\mp(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}][\pm(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_x^{3/2} + \phi_y] = 0$$

For curves within the flow field with the slope

$$\frac{dx}{dy} = \mp (\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2} \qquad (11)$$

this equation becomes

$$\frac{d}{dy}[\pm(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_x^{3/2} + \phi_y] = 0 \qquad (12a)$$

Hence, for these curves

$$\pm (\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_x^{3/2} + \phi_y = \text{const} \qquad (12b)$$

Curves whose slope is given by Eq. (11) are called characteristics. Eqs. (12a) and (12b) are called compatibility conditions for the characteristics. They express that along such curves ϕ_x and ϕ_y cannot be chosen independently. One speaks of left or right going characteristics to suggest their orientation with respect to the streamlines. (The positive sign in Eq. (11) refers to left going, the negative sign to right going characteristics.) In the method of characteristics one computes a net formed of such curves. The state at a grid point of such a net is readily found. Assume that one knows ϕ_x and ϕ_y at the points 1 and 2 of Figure 1. Points 1 and 2 must not lie on the same characteristics. Let point 3 be the point where two characteristics of different families through the points 1 and 2 intersect. Then one has

$$-(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_{x,3}^{3/2} + \phi_{y,3} = -(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_{x,1}^{3/2} + \phi_{y,1}$$

$$(13)$$

$$+(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_{x,3}^{3/2} + \phi_{y,3} = +(\gamma+1)^{1/2}a^{*-1/2}(2/3)\phi_{x,2}^{3/2} + \phi_{y,2}$$

(Second subscripts 1, 2 or 3 refer to the numbering of the points.) The right hand sides are known. Therefore, one can compute immediately, $\phi_{x,3}$ and $\phi_{y,3}$. After these quantities have been determined, one finds x_3 and y_3, by applying Eqs. (11) in difference form

$$\frac{x_3 - x_1}{y_3 - y_1} = (\gamma+1)^{1/2}a^{*-1/2}(1/2)(\phi_{x,3}^{1/2} + \phi_{x,1}^{1/2})$$

$$(14)$$

$$\frac{x_3 - x_2}{y_3 - y_2} = -(\gamma+1)^{1/2}a^{*-1/2}(1/2)(\phi_{x,3}^{1/2} + \phi_{x,2}^{1/2})$$

The flow field in the vicinity of the sonic line is regular; that is, ϕ can be developed in power series with respect to x and y (unless a singularity propagating along a

characteristic occurs at the point for which the development is to be carried out). However, the shape of the characteristics is singular, because of the power ${\phi_x}^{1/2}$ in Eq. (11). This can be taken into account when one forms Eqs. (14) in the segment of the characteristics next to the sonic line. If one chooses a point of the sonic line as origin of an x, y system and aligns the x axis with the local flow direction, then the lowest order term in the development of ϕ is given by

$$\phi = \text{const } xy$$

hence

$$\phi_x = \text{const } y$$

Therefore, for the characteristics in this vicinity

$$\frac{dx}{dy} = \mp(\gamma+1)^{1/2} a^{*-1/2} \text{const}^{1/2} y^{1/2}$$

and for the characteristics through the origin

$$x = \mp(\gamma+1)^{1/2} a^{*-1/2} \text{const}^{1/2} (2/3) y^{3/2}$$

Expressing y by ϕ_x, one obtains

$$x = \mp(\gamma+1)^{1/2} a^{*-1/2} \text{const}^{-1} (2/3) {\phi_x}^{3/2}$$

Hence, for two points of this characteristic

$$\frac{x_1 - x_2}{y_1 - y_2} = \mp(\gamma+1)^{1/2} a^{*-1/2} (2/3) \frac{\phi_{x_1}^{3/2} - \phi_{x_2}^{3/2}}{\phi_{x_1} - \phi_{x_2}} \tag{15}$$

Eqs. (14) and (15) are exact if, respectively, ${\phi_x}^{1/2}$ and ϕ_x and linear functions of y. Eq. (15) therefore, gives an acceptable approximation for the secant directions of the characteristics throughout the flow field, while Eqs. (14) becomes inaccurate in a neighborhood of the sonic line.

The sonic line is reached by only one downstream going characteristic (Fig. 2). The state at the sonic line is determined by the compatibility condition for this characteristic and the condition that at the sonic line $\phi_x = 0$.

Only one downstream going characteristic arrives at the surface of the profile (Fig. 3). For such a point, one has the condition for the slope of this characteristic, the compatibility condition, and as the boundary condition, the value of ϕ_y at the contour.

$$(\gamma+1)^{1/2} a^{*-1/2} (2/3) \phi_{x,2}^{3/2} + \phi_{y,2} = (\gamma+1)^{1/2} a^{*-1/2} (2/3) \phi_{x,2}^{3/2} + \phi_{y,1}$$

$$\frac{x_2 - x_1}{y_2 - y_1} = -(1/2)(\gamma+1)^{1/2} a^{*-1/2} \frac{2}{3} \frac{\phi_{x,2}^{3/2} - \phi_{x,1}^{3/2}}{\phi_{x,2} - \phi_{x,1}}$$

$$\phi_{y,2} = f(x_2) \qquad (16)$$

From these three equations, $\phi_{x,2}$, $\phi_{y,2}$ and x_2 are found by an iteration. To carry out the method of characteristics one also needs data along an initial line. They are given by an analytic expression valid for the entrance corner of the supersonic region. This will be discussed later. Busemann makes a special choice of the values of ϕ_y and ϕ_x which occur at the grid points of the characteristic net. This allows him to work with predetermined characteristic slopes. In the present context, it is probably preferable to forgo this simplification. One must, however, make sure that the differences in ϕ_y for adjacent points of the characteristic grid are not too large (because of accuracy requirements) and not too small (in order to avoid unnecessary work). The first condition will be the reason that one must start new downstream going characteristics at the surface of the profile.

To obtain the local changes $\Delta\phi_x$ and $\Delta\phi_y$ one might start with the linearized form of Eq. (7),

$$-(\gamma+1)a^{*-1}\phi_x\Delta\phi_{xx} - (\gamma+1)a^{*-1}\Delta\phi_x\phi_{xx} + \Delta\phi_{yy} = 0 \tag{17}$$

The characteristic conditions are obtained by rewriting this equation in the form

$$\pm(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}[\mp(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}]\Delta\phi_x$$

$$+[\mp(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}]\Delta\phi_y - (\gamma+1)a^{*-1}\phi_{xx}\Delta\phi_x = 0$$

Here all quantities except $\Delta\phi_x$ and $\Delta\phi_y$ are considered as known. The characteristics are again in line with the slope

$$dx/dy = \mp(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}$$

that is, they are identical with the characteristics of the original field. The compatibility conditions assume the form

$$\pm(\gamma+1)^{1/2}a^{*-1/2}\phi_x^{1/2}\frac{d}{dy}(\Delta\phi_x) + \frac{d}{dy}(\Delta\phi_y) - (\gamma+1)a^{*-1}\phi_{xx}\Delta\phi_x = 0 \tag{18}$$

The singularity which occurs in the shape of the characteristics and the factor $\phi_x^{1/2}$ which occurs in the last equation would make further discussions for the vicinity of the sonic line necessary.

It is simpler if one carries out the method of characteristics for the neighboring field as well as for the original field without introducing a linearization. At the beginning of Section II we defined the displacement of a point the flow field due to the superposition of a perturbation: the displacement vector is the difference in

the local vectors of points of the perturbed flow field and of the original flow field for which the state is the same. In applying the method of characteristics for the neighboring field it is best to compute these displacement vectors. The hodograph equation is linear, from this fact it follows that one obtains again an exact solution in the physical plane, even for the full potential equation, if one multiplies all displacement vectors by a constant factor. As far as the displacement vectors are concerned, this makes a limiting process to an infinitesimally small perturbation unnecessary. A limiting process is, however, necessary to determine from the displacement vectors the local changes $\Delta\phi_x$ and $\Delta\phi_y$. Let Δx and Δy be the components of the displacement vector. From the condition that at points of the original and of the perturbed field which correspond to each other the velocity vector is the same, one then obtains

$$\begin{aligned} \Delta\phi_x &= -\phi_{xx}\Delta x - \phi_{xy}\Delta y \\ \Delta\phi_y &= -\phi_{xy}\Delta x - \phi_{yy}\Delta y \end{aligned} \tag{19}$$

The derivatives ϕ_{xx}, ϕ_{xy} and ϕ_{yy} are taken from the basic flow field. The perturbations $\Delta\phi_x$ and $\Delta\phi_y$ are needed only at the current location of the sonic line and at the current location of the shock.

The method of characteristics which determines primarily the location of the grid points for a given state of the flow gives the displacement vectors directly. To start the procedure, one must use the displacement vectors along some initial line where they will be computed analytically from data at the entrance corner of the supersonic region.

If one has points 1 and 2 in the original field and points 1' and 2' in the neighboring field, then one obtains the point 3' in the neighboring field by drawing the parallel

to the characteristics 13 and 23 (see Figure 4). The displacement vector is then given by the line 33'. Actually it can be computed if only the displacement vectors 1,1' and 2,2' and the slopes of the characteristic 1,3 and 2,3 are known.

Let point 2 be the point of the original flow field where the downstream going characteristic through point 1 of the perturbed field which corresponds to point 1. For the displacement vector at point 2 the component d_n normal to the streamlines is known. Then one has as locus for point 2' the parallel through point 1' to the characteristic 1,2 and the parallel to the streamline at point 2 at a distance d_n. With these data one determines the component of the displacement d_t in streamwise direction at point 2. For this computation, only the displacement vector 1,1', the assumed displacement component d_n at point 2 and the slope of the characteristic 1,2 are needed.

At the profile contour one must express the fact that the local flow angle remains unchanged if one superimposes a perturbation. One has, in general, for corresponding points

$$\Delta\theta + (d\theta/dn)d_n + (d\theta/dt)d_t = 0$$

Therefore, at the profile contour

$$(d\theta/dn)d_n + (d\theta/dt)d_t = 0 \tag{20}$$

$d\theta/dt$ is the local curvature of the profile. $d\theta/dn$ is computed from the basic flow field, best probably from the data along the two characteristics which pass through the point of the contour in question. Let, temporarily, the y direction coincide with the normal to the streamlines (and to the profile contour). Then one has, by differentiation along the characteristics 1,2 and 1,3

$$(d\theta/dy)_{\ell,r} = d\theta/dn + d\theta/dt(dt/dy)_{\ell,r}$$

where $d\theta/dy$ and dt/dy is to be formed along the respective characteristics (subscripts ℓ or r refer to left or right going characteristics). These equations are formed for both characteristics at the point of the profile contour. The derivatives $(d\theta/dy)_\ell$ and $(d\theta/dy)_r$ for the point at the contour are found from the basic flow field. Making use of the fact that $(dt/dy)_\ell = -(dt/dy)_r$, one obtains

$$d\theta/dn = 1/2((d\theta/dy)_\ell + (d\theta/dy)_r) \tag{21}$$

of course, d_n and d_t (in Eq. (20)) must be expressed in terms of Δx and Δy. One has, for the simplified transonic equation

$$d_n = \Delta y \quad , \quad dt = \Delta x \tag{22}$$

Point 2' lies on the parallel to the characteristic 2 through point 1', and on the straight line defined by Eq. (20).

The flow field in the entrance corner to the supersonic region is determined analytically. At the sonic line, one has $\phi_x = 0$. Let the sonic line be given by $y = y_{sonic}(x)$.

$$\phi_{xx} + \phi_{xy}(dy_{sonic}/dx) = 0$$

Here, ϕ_{xy} is given by the boundary condition at the profile. In geometric terms one has

$$a^{*-1}\phi_{xy} = R^{-1} \tag{23}$$

where R is the radius of curvature at the point under consideration. Further information is provided by the observation that one can obtain the development of the potential for the vicinity of a point of the contour if the velocity vector at the contour is given, for the contour is not a

characteristic line. In the present context, the result is immediately obvious. One obtains from the simplified transonic equation for the entrance corner of the supersonic region

$$\phi_{yy} = 0$$

One then obtains in a sufficiently small neighborhood a coordinate system whose origin lies at the entrance corner of the supersonic region

$$\begin{aligned} \phi_x &= \phi_{xy}(-(dy_{sonic}/dx)x + y) \\ \phi_y &= \phi_y(0,0) + \phi_{xy}x \end{aligned} \tag{24}$$

To start the procedure one will compute the value of ϕ_y at point 1 of Figure 7; $\phi_x = 0$ because this is a point of the sonic line.

Next, one introduces a perturbation by assuming that for y = 0 (that is along the contour) ϕ_x is given by

$$\phi_x(x,0) = \Delta\phi_x(0,0) + \phi_{xx}(0,0)x + \Delta\phi_{xx}(0,0)x$$

Since ϕ_{xy} is determined by the profile shape, one has in the changed flow field

$$\phi_x(x,y) + \Delta\phi_x(x,y) = \Delta\phi_x(0,0) + \phi_{xx}(0,0)x + \Delta\phi_{xx}(0,0)x + \phi_{xy}(0,0)y$$

Thus, for the local change

$$\Delta\phi_x(x,y) = \Delta\phi_x(0,0) + \Delta\phi_{xx}(0,0)x \tag{25}$$

Now, we invoke the differential equation (for x = 0, y = 0)

$$-(\gamma+1)a^{*-1}\Delta\phi_x(0,0)[\phi_{xx}(0,0) + \Delta\phi_{xx}(0,0)] + \Delta\phi_{yy}(0,0) = 0$$

Hence, with a linearization

$$\Delta\phi_{yy}(0,0) = (\gamma+1)a^{*-1}\Delta\phi_x(0,0)\phi_{xx}(0,0) \tag{26}$$

and

$$\Delta\phi_y(x,y) = (\gamma+1)a^{*-1}\Delta\phi_x(0,0)\phi_{xx}(0,0)y \tag{27}$$

To determine from these results the displacement vector (components Δx and Δy) in the flow field, we express the fact that at corresponding points the values of ϕ_x and ϕ_y are the same.

$$\phi_{xx}(0,0)x + \phi_{xy}(0,0)y = \Delta\phi_x(0,0) + \phi_{xx}(0,0)(x+\Delta x) + \Delta\phi_{xx}(0,0)x$$
$$+ \phi_{x\ y}(0,0)(y + \Delta y)$$

$$\phi_{xy}(0,0)x = \phi_{xy}(0,0)(x + \Delta x) + \Delta\phi_{yy}(0,0)y$$

Hence

$$\Delta x(x,y) = -\frac{\Delta\phi_{yy}(0,0)}{\phi_{xy}(0,0)}\,y \tag{28}$$

$$\Delta y(x,y) = \frac{1}{\phi_{xy}(0,0)}\left[-\Delta\phi_x(0,0) - \Delta\phi_{xx}(0,0)x + \frac{\phi_{xx}(0,0)\Delta\phi_{yy}(0,0)}{\phi_{xy}(0,0)}\,y\right] \tag{29}$$

Here $\Delta\phi_{yy}(0,0)$ is expressed by Eq. (26). So far unknown are the assumed perturbations at the contour $\Delta\phi_x(0,0)$ and $\Delta\phi_{xx}(0,0)$. The purpose of these computations is the evaluation of the kernel $K(s,t)$ in Eq. (6). One chooses $d_n(t) = 0$, up to a point $t = \tau$ and $d_n(t) = 1$, from there on; in other words,

$d_n'(t)$ and $\delta(t - \delta)$. For this choice one computes $d_t(s,t)$. This function depends, of course, upon the value of τ. Then, from Eq. (6)

$$d_t(s,\tau) = K(s,\tau) \tag{30}$$

These computations must be carried out for a sufficient number of values of t and s so that, if necessary, intermediate values of K(s,t) can be obtained by interpolation.

If the point $t = \tau$ lies at the entrance corner of the supersonic region, ($x = 0$, $y = 0$ in Eq. (29)) then one obtains

$$\Delta y(0,0) = -\Delta\phi_x(0,0)/\phi_{xy}(0,0)$$

Hence $\Delta\phi_x(0,0) = -\phi_{xy}(0,0)\ \Delta y(0,0)$ (31)

For the evalution of K(s,t) described above, one has $d_n(\tau) = \Delta y = 1$ also for other points of the sonic line. Therefore, from Eq. (29)

$$1 = \frac{1}{\phi_{xy}(0,0)}[-\Delta\phi_x(0,0) - (\Delta\phi_{xx}(0,0) - \frac{\phi_{xx}(0,0)\Delta\phi_{yy}(0,0)}{\phi_{xy}(0,0)}\ \frac{dy_{sonic}}{dx})x]$$

Hence, with Eqs. (31) and (26)

$$\Delta\phi_{xx}(0,0) = -(\gamma+1)a^{*-1}\phi_{xx}(0,0)^2\ \frac{dy_{sonic}}{dx}\ \Delta y(0,0)$$

This allows one to compute $\Delta x(x,y)$ and $\Delta y(x,y)$, for the point at which the method of characteristics starts (points 1 in Figure 7) by means of Eqs. (25) and (27).

The remaining discussions address themselves to the character of the kernel K(s,t) for points at the sonic line, other than the initial point. The computation described above introduces at the station $t = \tau$ a jump of $d_n(t)$, one therefore expects that $d_t(s)$ will be singular for $s = t$.

It is desirable to know, at least qualitatively, the character of these singularities, for the numerical evaluation gives only pointwise data. For this purpose it is best to use the simplified hodograph equation, which arises from the simplified transonic equation by the Legendre transformation. To make the report self-contained, the fundamental derivations are repeated (see for instance Ref. 2). One starts from

$$-(\gamma+1)\phi_x\phi_{xx} + \phi_{yy} = 0$$

(for simplicity we have set a* = 0)
and sets

$$\phi_x = \eta \quad , \quad \phi_y = 0 \tag{33}$$

$$\phi(x,y) = -\Phi(\eta,\theta) + x\eta + y\theta \tag{34}$$

Then

$$\phi_x = -\Phi_\eta \frac{\partial\eta}{\partial x} - \Phi_\theta \frac{\partial\theta}{\partial x} + \eta + x \frac{\partial\eta}{\partial x} + y \frac{\partial\theta}{\partial x}$$

$$\phi_y = -\Phi_\eta \frac{\partial\eta}{\partial y} - \Phi_\theta \frac{\partial\theta}{\partial y} + x \frac{\partial\eta}{\partial y} + \theta + y \frac{\partial\theta}{\partial y}$$

Hence with Eq. (33)

$$0 = \frac{\partial\eta}{\partial x}(-\Phi_\eta + x) + \frac{\partial\theta}{\partial x}(-\Phi_\theta + y)$$

$$0 = \frac{\partial\eta}{\partial y}(-\Phi_\eta + x) + \frac{\partial\theta}{\partial y}(-\Phi_\theta + y)$$

and

$$x = \Phi_\eta \quad , \quad y = \Phi_\theta \tag{35}$$

The derivatives ϕ_{xx}, ϕ_{xy} and ϕ_{yy} are found from the identities

$$\eta = \phi_x(x(\eta,\theta),y(\eta,\theta))$$

and

$$\theta = \phi_y(x(\eta,\theta),y(\eta,\theta))$$

One obtains by differentiating the first one with respect to η and θ

$$1 = \phi_{xx}\frac{\partial x}{\partial \eta} + \phi_{xy}\frac{\partial y}{\partial \eta}$$

$$0 = \phi_{xx}\frac{\partial x}{\partial \theta} + \phi_{xy}\frac{\partial y}{\partial \theta}$$

Hence

$$\begin{aligned} \phi_{xx} &= D^{-1}\frac{\partial y}{\partial \theta} = D^{-1}\,\Phi_{\theta\theta} \\ \phi_{xy} &= -D^{-1}\frac{\partial x}{\partial \theta} = -D^{-1}\,\Phi_{\eta\theta} \end{aligned} \tag{36}$$

with

$$D = \left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \eta}\right) = \Phi_{\eta\eta}\Phi_{\theta\theta} \quad \Phi^2_{\eta\theta} \tag{37}$$

Similarly, from the second equation

$$\phi_{yy} = D^{-1}\Phi_{\eta\eta} \tag{38}$$

Hence, from Eq. (8)

$$-(\gamma+1)\eta\Phi_{\theta\theta} + \Phi_{\eta\eta} = 0 \tag{39}$$

The lowest order approximation to the basic flow in the vicinity of a point of the sonic line is given by

$$\Phi(\eta,\theta) = R[\eta\theta + (\frac{\gamma+1}{6}\eta^3 + \frac{\theta^2}{2})\operatorname{tg}\beta]$$

Hence

$$x = \Phi_\eta = R[\theta + \frac{\gamma+1}{2}\eta^2 \operatorname{tg}\beta]$$

$$y = \Phi_\theta = R[\eta + \theta\operatorname{tg}\beta] \tag{40}$$

This expression satisfies

$$\phi_{xy} = -\frac{\Phi_{\eta\theta}}{D} = \frac{-R}{\operatorname{tg}^2\beta R^2(\gamma+1)\eta - R^2}$$

Hence, for η small

$$\phi_{xy} = R^{-1}$$

in accordance with Eq. (2). The sonic line is given by $\eta = 0$. There one has

$$x = R\theta$$

$$y = R\theta\operatorname{tg}\beta$$

In accordance with the definition following Eq. (2), one obtains, indeed,

$$dy_{sonic}/dx = \operatorname{tg}\beta$$

If one superimposes to Φ a solution $\Phi^{(1)}$ then, $\Phi_\eta^{(1)}$ and $\Phi_\theta^{(1)}$ are the x and y components of the displacement vector. We consider now a solution $\Phi^{(1)}$ for which $\Phi_\theta^{(1)}$ evaluated at the sonic line ($\eta = 0$) has a jump as one proceeds from positive to negative values of θ and is constant otherwise. The choice of the signs is made in accordance with the behavior of the flow field on the upper side of a profile. At the sonic line the angle θ decreases as one proceeds in the downstream direction and R is negative.

Particular solutions of this character have the form

$$\Phi(\eta,\theta) = \eta^{3/2} f(\zeta) \tag{41}$$

where

$$\zeta = \frac{9}{4}(\gamma + 1)^{-1} \frac{\theta^2}{\eta^3} \tag{42}$$

The characteristics of Eq. (39) are recognized if one rewrites it in the following manner

$$\pm(\gamma+1)^{1/2}\eta^{1/2}[\mp(\gamma+1)^{1/2}\eta^{1/2} \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\eta}]\Phi_\theta$$

$$+ [\mp(\gamma+1)^{1/2}\eta^{1/2}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\eta}]\Phi_\eta = 0$$

They are given by

$$\frac{d\theta}{d\eta} = \mp(\gamma+1)^{1/2}\eta^{1/2}$$

or

$$\theta \pm (\gamma+1)^{1/2}(2/3)\eta^{3/2} = \text{const}$$

The lines $\zeta = 1$ are the characteristic through the origin. One solution for which Φ_θ = const along a line ζ = const is obviously given by

$$\Phi = (3/2)(\gamma+1)^{1/2}\theta$$

which leads to

$$f = \zeta^{1/2} \tag{43}$$

but it is not suitable for the present purpose. One obtains, by substituting Eq. (41) into Eq. (39)

$$\zeta(\zeta-1)f'' + [\frac{\zeta}{3} - \frac{1}{2}]f' + \frac{1}{12}f = 0 \tag{44}$$

This is indeed satisfied by Eq. (43). Eq. (44) is reduced to a first order equation by means of Eq. (43).

$$f = \zeta^{1/2} g(\zeta)$$

$$\zeta(\zeta-1)g'' + (\frac{4}{3}\zeta - \frac{3}{2})g' = 0$$

Hence

$$\frac{g''}{g'} = \frac{\frac{3}{2} - \frac{4}{3}\zeta}{\zeta(\zeta-1)} = -\frac{3}{2}\frac{1}{\zeta} + \frac{1}{6}\frac{1}{\zeta - 1}$$

Then, except for a factor

$$g' = \zeta^{-3/2}(\zeta - 1)^{1/6}$$

$$g(\zeta) = \int_{\tilde{\zeta}=1}^{\zeta} \tilde{\zeta}^{-3/2} (\tilde{\zeta}-1)^{1/6} d\tilde{\zeta}$$

With this choice of the constant of integration one obtains $g = 0$, $g' = 0$ for $\zeta = 1$. This gives an admissible transition along the characteristic $\zeta = 1$ (Fig. 8 in the second quadrant) from the solution $\phi^{(1)} \equiv 0$, to the expression $\Phi = \eta^{3/2}\sqrt{\zeta}g(\zeta)$ for Φ as well as its first derivative vanish along this line. The sonic line is given by $\eta = 0$, $\zeta = \infty$. One has

$$g(\infty) = \int_{\tilde{\zeta}=1}^{\infty} \tilde{\zeta}^{3/2} (\tilde{\zeta}-1)^{1/6} d\tilde{\zeta}$$

This constant can be evaluated in terms of tabulated functions. One obtains by setting

$$\tilde{\zeta} = \zeta^{-1}$$

$$g(\infty) = \int_0^1 \zeta^{1/3 - 1} (1-\zeta)^{7/6 - 1} d\zeta = B(\frac{1}{3}, \frac{7}{6})$$

where B is the Eulerian Beta-function. Hence

$$g(\infty) = \frac{\Gamma(\frac{1}{3})\ \Gamma(\frac{7}{6})}{\Gamma(\frac{3}{2})}$$

One has for ξ small

$$g(\xi^{-1}) = \int_{\xi}^{1} \xi^{-2/3}(1-\xi)^{1/6}d\xi = \int_{0}^{1} - \int_{0}^{\xi} \xi^{-2/3}(1-\xi)^{1/6}d\xi$$

$$= \frac{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}{\Gamma(\frac{3}{2})} - 3\xi^{1/3} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}{\Gamma(\frac{3}{2})} - 3\zeta^{-1/3}$$

Hence for the vicinity of the line $\eta = 0$ ($\zeta = \infty$)

$$\Phi^{(1)} = \eta^{3/2}\ \zeta^{1/2}\ g(\zeta)$$

$$= \frac{3}{2}(\gamma+1)^{-1/2}\ \frac{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}{\Gamma(\frac{3}{2})}\ \theta - 3^{4/3}\ 2^{-1/3}(\gamma+1)^{-1/6}\theta^{1/3}\eta$$

Therefore, for $\eta = 0$

$$\Phi_{\eta}^{(1)} = -3^{4/3}\ 2^{-1/3}(\gamma+1)^{-1/6}\ \theta^{1/3}$$

$$\Phi_{\theta}^{(1)} = \frac{3}{2}(\gamma+1)^{-1/2}\ \frac{\Gamma(\frac{1}{3})\ \Gamma(\frac{7}{6})}{\Gamma(\frac{3}{2})}$$

The displacement in the y direction is given by $\Phi_{\theta}^{(1)}$. In the lowest approximation this displacement is constant at the boundary $\eta = 0$ of the shaded region of Fig. 8, while it is zero for θ positive. The component in the streamline direction of the displacement vector is then given by

$$\phi_\eta^{(1)} = -\phi_\theta^{(1)} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{3})\ \Gamma(\frac{7}{6})} (\gamma+1)^{1/3}\ 3^{1/3}\ 2^{2/3}\ \theta^{1/3}$$

Now we had in the basic flow, in the lowest approximation, according to Eq. (40)

$$\theta = \frac{x}{R}$$

Hence, for points of the sonic line immediately downstream of the point where the jump occurs

$$d_t = -d_n \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{3})\ \Gamma(\frac{7}{6})} (\gamma+1)^{1/3}\ 3^{1/3}\ 2^{2/3}\ R^{-1/3}\ x^{1/3}$$

This equation displays the character of $d_t(x)$ taken along the sonic line for the vicinity of a point where $d_n(s)$ has a jump. Important for the purpose of interpolation is the factor $x^{1/3}$. Sometimes it may not be necessary to determine the coefficient by which this factor is multiplied. The qualitative behavior of some quantities which are needed for subsonic computations is shown in Fig. 9.

The singularity generated by a jump of $d_n(x)$ will propagate in the downstream direction along the characteristic which starts at the point where the jump occurs. It is reflected at the surface of the profile. A singularity of the same character will occur where this reflected wave reaches the sonic line. This wave is reflected again. These discussions show the character of the singularities which the kernel K(s,t) in Eq. (6) will have. As was mentioned above, the numerical evaluation gives K(s,t) pointwise.

In deriving boundary conditions for the subsonic region from the Eq. (6), one must discretize the relations for $\Delta\phi_x$ and $\Delta\phi_y$ and $d_n(s)$ in a form which is suitable for the computation of the subsonic region. This means that one must approximate these data in a form analogous to the expressions for $\Delta\phi_x$ and $\Delta\phi_y$ which appears in the subsonic finite element formulation. Details will, of course, depend upon the character of the shape functions chosen in the subsonic region.

This provides, at least in principle, the system of formulae needed to carry out the procedure sketched in Section II.

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2. Guderley, K. G., The Theory of Transonic Flow, Pergamon Press 1962, page 74.

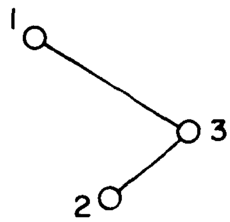

Figure 1. Characteristics Within the Field.

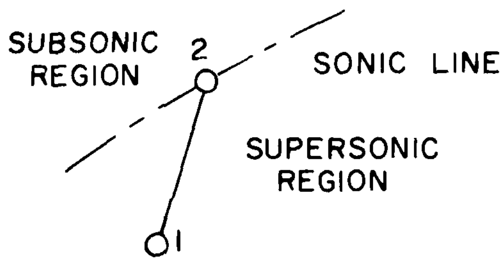

Figure 2. Characteristic Terminating at the Sonic Line.

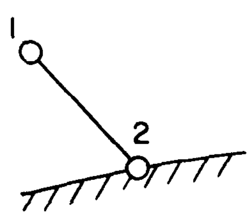

Figure 3. Characteristic Terminating at the Profile Contour.

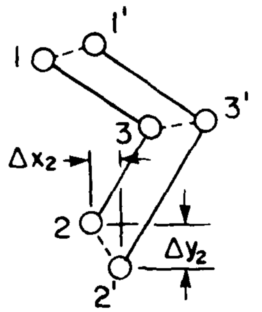

Figure 4. Characteristics Connecting Points of a Basic Field (Points 1, 2 and 3) and of a Perturbed Field (Points 1',2' and 3').

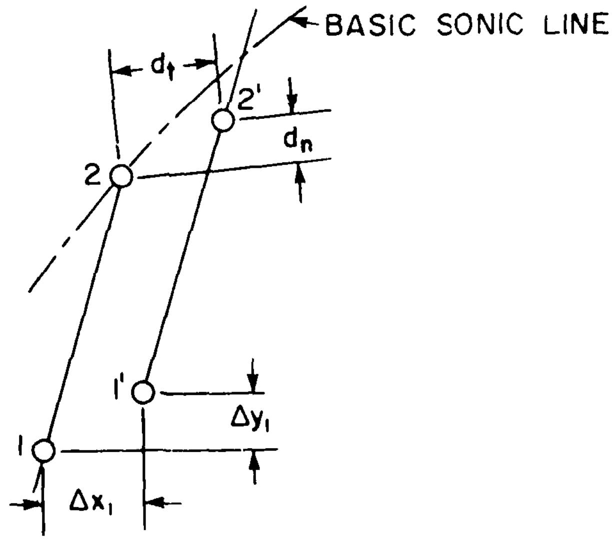

Figure 5. Computation of the Displacement Vector of a Point of the Sonic Line. (Δx_1, Δy_1 and d_n are known, d_t is to be determined)

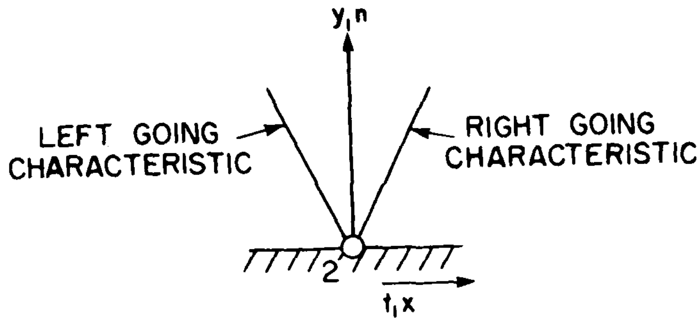

Figure 6. Sketch Illustrating the Relation Between $\frac{d\theta}{dy}_{\ell}$, $\frac{d\theta}{dy}_{r}$, $\frac{d\theta}{dn}$ and $\frac{d\theta}{dt}$.

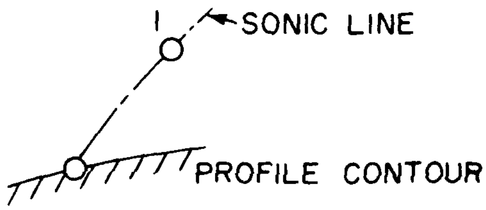

Figure 7. Starting Point (1) for the Method of Characteristics.

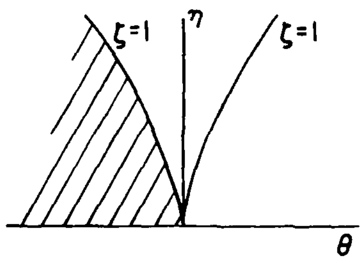

Figure 8. *Hodograph* (η, θ plane). The Curves $\zeta = 1$ are the Characteristics through the Origin. The Shaded Part is Affected by a Jump of d_n.

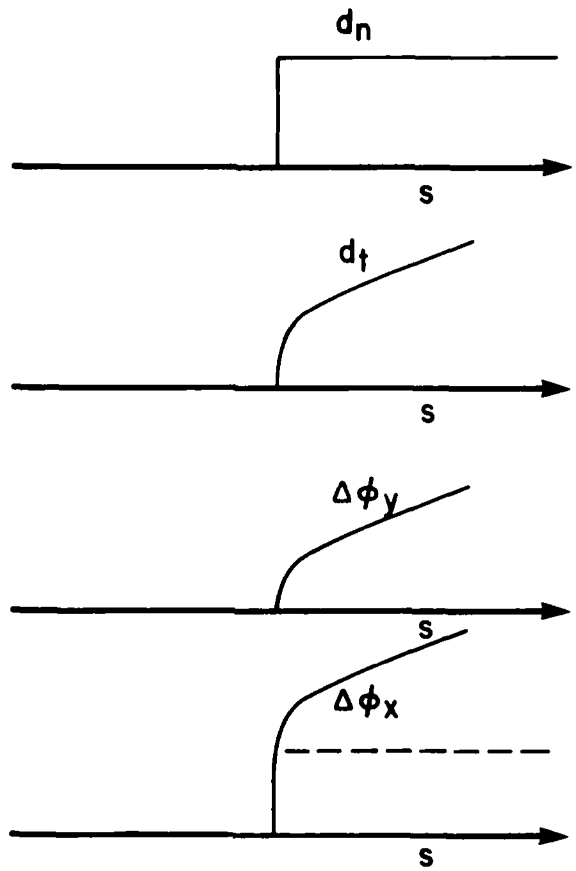

Figure 9. Qualitative Behavior of d_t, $\Delta\phi_y$ and $\Delta\phi_x$ at a Point of the Sonic Line where d_n Jumps from Zero to a Finite Value.

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